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A NUMERICAL TECHNIQUE FOR THREE-DIMENSIONAL COMPRESSIBLE BOUNDARY LAYERS

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SUMMARY

The non-linear two-point boundary value problem for three-dimensional compressible boundary layers is solved through the application of a boundary value technique for a range of parameters characterizing the nature of stagnation point flows. The analytical boundary conditions, at infinity, are applied at the edge of the computational mesh with iterations on the size of the domain. The solutions obtained show excellent agreement with the established similarity solutions for three-dimensional flows. The present method has the potential advantage of yielding the wall values of f''_w, g''_w and θ'_w as a part of the solution, contrary to the previously used 'shooting' methods. The algorithm is computationally simple and numerically stable and extremely suitable for engineering design applications.

KEY WORDS Compressible Boundary Layer Stagnation Point Flow Boundary Value Problem

INTRODUCTION

The analysis of compressible boundary layers at a general three dimensional stagnation point has been motivated by the basic nature of the boundary-layer flow at such points, by the exact applicability there of similarity solutions, and by their relevance to the leading edge and nose regions of bodies in high speed flight. The solution is of immense importance in the design of thermal protection systems for launch vehicles, as well as for spacecraft re-entering planetary atmospheres at hypersonic speeds.

Assuming viscosity to be a linear function of the temperature and using a two component stream function (S_1, S_2) with the appropriate co-ordinate transformation, the pertinent boundary-layer equations reduce to the system of ordinary differential equations (following Reshotoko[1]):

$$f''' + (f + Kg)f'' + \left[1 + \left(\frac{H_w}{H_0} - 1\right)(1 - \theta) - f'^2\right] = 0 \tag{1}$$

$$g''' + (f + Kg)g'' + K\left[1 + \left(\frac{H_w}{H_0} - 1\right)(1 - \theta) - g'^2\right] = 0 \tag{2}$$

$$\theta'' + Pr(f + Kg)\theta' = 0 \tag{3}$$

where f and g are the stream functions for the similarity solutions and θ is the dimensionless enthalpy. The prime denotes differentiation with respect to the similarity variable, η. The parameters used in equations (1)–(3) are defined in the nomenclature.

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The boundary conditions for these equations are:

$$\eta = 0 : f = f' = g = g' = \theta = 0 \tag{4}$$

$$\eta \to \infty : f' = g' = \theta = 1 \tag{5}$$

The parameter $\theta'_w = \theta'(0)$ is of prime interest in thermal design since it relates to the wall heat transfer rate.

Equations (1)–(3) with the boundary conditions (4) and (5) constitute an eighth order non-linear system of ordinary differential equations with three arbitrary parameters and define a difficult non-linear two point boundary value problem. The above system of equations and its variants have been solved for a wide range of parameters by Poots,[2] Vimala and Nath,[3], Nath and Meena,[4] Narayana and Ramamoorthy,[5] Libby,[6–8] Wortman[9,10] and Balu.[11] In Reference 2, the solution was obtained by solving an initial value problem using fourth order Runge–Kutta integration and iteratively adjusting the unknown wall values of f'', g'' and θ' until the boundary conditions of equation (5) were satisfied, a 3-space application of the well-known 'shooting' technique. In addition to small $\Delta\eta$, the main difficulty was in selecting a set of initial values for every set of parameters, Pr, H_w/H_0 and K. References 3–5 used parametric differentiation of Ruppert and Landahl[12] and also considered the effect of wall blowing ($f'(0) \neq 0$). References 6–8 used a quasilinearization procedure coupled with asymptotic solutions which involved considerable computational effort. References 9 and 10 used a functional approach and Reference 11 used Keller's method for the solution of the problem. The turbomachinery applications of the similarity relations have been widely studied using the Runge–Kutta shooting method by Wang *et al.*[13] and for an oscillating main stream by Gorla.[14,15]

In most of the previous analyses, the major source of difficulty has been the determination of the proper values of f'', g'' and θ' at the wall to ensure an asymptotic approach of the velocities f' and g', and θ to unity at infinity, the well known matching conditions of the viscous (near wall) solution to the inviscid solution.

The aim of this paper is to extend a simple, direct, finite-difference method used in the solution of two dimensional incompressible stagnation point flows,[16] to three dimensional compressible flows with possible direct applications to the annulus wall boundary layer regions for rotor tip leakage analyses[17] and subsequent correlation with heat transfer measurements.[18] The method uses Richardson's[19] general concept of using the infinity boundary conditions at the extremes of the mesh. Neglecting the details of the 'far' downstream flow did not lead to catastrophic instabilities being propagated upstream from the outflow boundaries. This concept has been demonstrated to be realistic if special localized grids[16] are used to account for the boundary conditions at infinity in the discretization procedure with a finite computational domain with the iterations on the size of the domain. The following sections describe the method and present some of the results obtained. Comparisons with other available solutions show excellent agreement.

METHOD OF SOLUTION

Reformulation of the governing equations

Equations (1) and (2) can be rewritten as a system of first and second order equations by defining new variables ϕ and ψ as

$$f' = \Psi; \quad g' = \phi \tag{6}$$

$$\Psi'' + (f + Kg)\Psi' + \left[1 + \left(\frac{H_w}{H_0} - 1\right)(1 - \theta) - \Psi^2\right] = 0 \tag{7}$$

$$\phi'' + (f + Kg)\phi' + K\left[1 + \left(\frac{H_w}{H_0} - 1\right)(1-\theta) - \phi^2\right] = 0 \tag{8}$$

The boundary conditions in equations (4) and (5) in terms of the new variables ϕ and Ψ are

$$f(0) = g(0) = 0 \tag{9}$$

$$\Psi(0) = 0, \lim_{\eta \to \infty} \Psi(\eta) = 1{\cdot}0 \tag{10}$$

$$\phi(0) = 0, \lim_{\eta \to \infty} \phi(\eta) = 1{\cdot}0 \tag{11}$$

corresponding to equations (6), (7) and (8), respectively.

Finite difference approximation

A logarithmically varying grid was used within the laminar sublayer and the finite difference equations are obtained for a non-uniform grid with a dense mesh system near the wall region.

Equation (6) is solved as an initial value problem. Within the laminar sublayer ($i = 1, j$), the potential advantage of using a logarithmically varying grid was used to the fullest possible extent to accelerate the computational time using the numerical quadrature formula presented in Reference 20. For the non-uniform grid spacing outside the laminar sublayer, the trapezoidal rule was used, such that

$$f_{i+1} = f_i + \tfrac{1}{2}(\Psi_i + \Psi_{i+1})(\eta_{i+1} - \eta_i); \quad f(0) = f_1 = 0; \quad i = j, j+1, \ldots, n+1 \tag{12a}$$

$$g_{i+1} = g_i + \tfrac{1}{2}(\phi_i + \phi_{i+1})(\eta_{i+1} - \eta_i); \quad g(0) = g_1 = 0; \quad i = j, j+1, \ldots, n+1 \tag{12b}$$

with errors

$$E_g = \frac{(\eta_{i+1} - \eta_i)^3}{12} \frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2}\bigg|_{\eta=\xi} \quad \text{and} \quad E_f = \frac{(\eta_{i+1} - \eta_i)^3}{12} \frac{\mathrm{d}^2\Psi}{\mathrm{d}\eta^2}\bigg|_{\eta=\xi} \tag{12c}$$

where the domain $(0, \eta_\infty)$ is divided into n unequal parts and j represents the node approximately at the edge of the laminar sublayer or the last node number of the logarithmically varying grid. For more accuracy it is possible to use the higher order quadrature formula of McNammee[21] for unequal intervals.

Equations (7), (8) and (3) are solved as two point boundary value problems using optimized successive over relaxation (OSOR). The finite difference approximations of equations (7), (8) and (3) at any interior node are:

$$\Psi_i^{k+1} = \Psi_i^k + \omega\left\{\frac{2[h_1\Psi_{i+1}^k + h_2\Psi_{i-1}^{k+1}] + [f_i + Kg_i][h_1^2\Psi_{i+1}^k - h_2^2\Psi_{i-1}^{k+1}] + \left[1 + \left(\frac{H_w}{H_0} - 1\right)(1-\theta_i)\right][h_1h_2][h_1 + h_2]}{(h_1 + h_2)[2 + (f_i + Kg_i)(h_1 - h_2) + h_1h_2\Psi_i^k]} - \Psi_i^k\right\} \tag{13}$$

$$\phi_i^{k+1} = \phi_i^k + \omega\left\{\frac{2[h_1\phi_{i+1}^k + h_2\phi_{i-1}^{k+1}] + [f_i + Kg_i][h_1^2\phi_{i+1}^k - h_2^2\phi_{i-1}^{k+1}] + K\left[1 + \left(\frac{H_w}{H_0} - 1\right)(1-\theta_i)\right](h_1h_2)(h_1 + h_2)}{(h_1 + h_2)[2 + (f_i + Kg_i)(h_1 - h_2) + Kh_1h_2\phi_j^k]} - \phi_i^k\right\} \tag{14}$$

$$\theta_i^{k+1} = \theta_i^k + \omega\left\{\frac{2[h_1\theta_{i+1}^k + h_2\theta_{i-1}^{k+1}] + [f_i + Kg_i]Pr[h_1^2\theta_{i+1}^k - h_2^2\theta_{i-1}^{k+1}]}{(h_1 + h_2)[2 + (f_i + Kg_i)Pr(h_1 - h_2)]} - \theta_i^k\right\}, \quad i = 2, 3, \ldots, n \tag{15}$$

$$\Psi_1 = \phi_1 = \theta_1 = 0 \quad \text{and} \quad \Psi_{n+1} = \phi_{n+1} = \theta_{n+1} = 1 \tag{16}$$

where h_1 and h_2 and defined in the nomenclature.

Iteration sequence

The method of solution is based on the selection of any arbitrary finite value for the end of the computational mesh (η_{∞_1}) and solving the finite difference equations using arbitrary linear initial approximations for f, g, θ, ϕ and Ψ. The iteration scheme is illustrated schematically in Figure 1 and is described briefly herein.

The values of f are computed at all interior nodes using the quadrature formula of Reference 20 in the laminar sublayer and equation (12) outside the sublayer. The values of Ψ are then obtained by solving equation (13) with an optimized over-relaxation parameter, $\omega = 1{\cdot}25$. In the first few

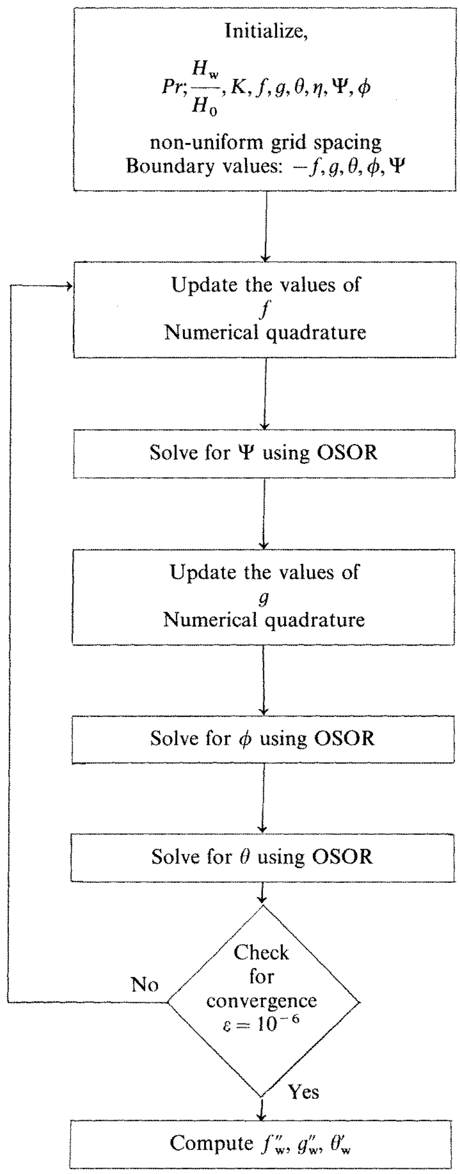

Figure 1 Schematic diagram of the iteration scheme

outer loops the number of relaxation passes are limited to 11. Numerical quadrature is used to compute g in a similar manner as f. ϕ and θ are then solved by over-relaxation using equations (14) and (15), respectively, proceeding on similar lines as the iterations for Ψ. If the maximum absolute value of f, g, Ψ, ϕ and θ between two outer loop iterations does not exceed the value of the convergence parameter ε (10^{-6}); the iterations are assumed to have converged. Numerical experiments indicate that the stability and the rate of convergence of the difference equations are not strongly dependent on the choice of the initial approximations of the five unknowns.

Doubling the size of the computational domain $(0, 2\eta_{\infty_1})$, the difference equations are solved again in a similar manner as stated above using the same step size variations of the non-uniform grid. If the two solutions of f, g, θ, ϕ and Ψ so obtained for $(0, \eta_{\infty_1})$ and $(0, 2\eta_{\infty_1})$ do not differ by more than five parts in one thousand, then the choice of $\eta_\infty = \eta_{\infty_1}$ is considered to be satisfactory and the values of f''_w, g''_w and θ'_w are computed. If η_∞ is unreasonably small, the solution oscillates in a diverging pattern or has a tendency to overshoot or undershoot in the vicinity of the infinity boundary condition. This behaviour was also noted in the case of two dimensional incompressible flows.[16]

NUMERICAL RESULTS AND DISCUSSIONS

The method described above was used to obtain the solution for several combinations of $K = -0{\cdot}5, 0, 0{\cdot}5, 1$; $H_w/H_0 = 0, 1, 2$ and $Pr = 0{\cdot}7, 1, 2$. The size of the mesh was varied between $\eta_\infty = 4$ and $\eta_\infty = 8$ and typically consisted of 25 to 65 nodes depending on the described accuracy. The step sizes investigated varied as $\Delta\eta = 0{\cdot}001, 0{\cdot}025, 0{\cdot}05, 0{\cdot}1, 0{\cdot}25$ and $0{\cdot}5$ near the wall. The effects of step size were not found to be critical in the convergence of the discretized system of equations. Table I illustrates the method involved in the mode of selection for the appropriate value of η_∞ for $K = 0{\cdot}5$, $H_w/H_0 = 2{\cdot}0$ and $Pr = 1{\cdot}0$.

Table II summarizes the results obtained in the form of f''_w, g''_w and θ'_w with comparisons with the values published in open literature for several different combinations of K, H_w/H_0 and Pr. All the calculations were performed with $h_w = 0{\cdot}025$, $\eta_\infty = 4{\cdot}0$, $8{\cdot}0$ and $\omega = 1{\cdot}25$ and the results show excellent agreement with those of Reshotoko.[1]

CONCLUDING REMARKS

The non-linear two point boundary value problem posed by equations (1)–(3) is solved successfully through the application of a boundary value technique for a range of K, H_w/H_0 and Pr, the parameters characterizing the nature of stagnation point flows. It is evident from the solution that there is good agreement between the results of the present investigation and those of References 1 and 2. The present method has the potential advantage of yielding the wall values of f''_w, g''_w and θ'_w as a part of the solution with the iteration being on the size of the computational domain contrary to previously used 'shooting' methods. Whereas 'shooting' methods are realistic for simpler applications (e.g. Falkner–Skan) their use as a 3-space application is very tedious and unrealistic especially where solutions of engineering accuracy are sufficient and the problem has to be solved over a wide range of parameters (e.g. design optimization). The algorithm is computationally simple, numerically stable if η_∞ is selected 'far' enough from the wall and extremely rapid for obtaining physically meaningful results.

It is possible to anticipate that the present method could be used as a springboard for further studies on three dimensional compressible flows with suction or blowing and for unsteady flows.

Table I. Variation of f, g and θ with η for different step sizes. $K = 0{\cdot}5$, $H_w/H_0 = 2{\cdot}0$, $Pr = 1{\cdot}0$

η	0·0	0·5	1·0	2·0	3·0	4·0	6·0	8·0	$h_w^{\dagger}$	Number of iterations	f''_w	g''_w	θ'_w	η_∞
f	0·0	0·1773	0·5754	1·5558	2·5574	3·5575	—	—	0·0125	60	1·7341	1·2737	0·7074	4·0
f	0·0	0·1729	0·5685	1.5476	2.5489	3.549	—	—	0·25	10	1·7146	1.2693	0·7108	4·0
f	0·0	0·1773	0·5754	1·5558	2·5574	3·5575	5·5575	7·5575	0·0125	82	1·7341	1·2737	0·7074	8·0
f'	0·0	0·6360	0·9110	1·003	1·0003	1·000	—	—	0·0125	60	1·7341	1·2737	0·7074	4·0
f'	0·0	0·6365	0·9114	1·003	1·0002	1·000	—	—	0·25	10	1·7146	1·2693	0·7108	4·0
f'	0·0	0·6361	0·9110	1·003	1·0003	1·0000	1·0	1·0	0·0125	82	1·7341	1·2737	0·7074	8·0
g	0·0	0·139	0·4801	1·4117	2·4089	3·4089	—	—	0·0125	60	1·7341	1·2737	0·7074	4·0
g	0·0	0·1367	0·4762	1·407	2·4042	3·4042	—	—	0·25	10	1·7146	1·2693	0·7108	4·0
g	0·0	0·139	0·4801	1·4117	2·4089	3·4089	5·4089	7·4089	0·0125	82	1·7341	1·2737	0·7074	8·0
g'	0·0	0·5170	0·8145	0·9889	0·9999	1·0	—	—	0·0125	60	1·7341	1·2737	0·7074	4·0
g'	0·0	0·5175	0·8156	0·9896	0·9999	1·0	—	—	0·25	10	1·7146	1·2693	0·7108	4·0
g'	0·0	0·517	0·8145	0·9889	0·9999	1·0000	1·0	1·0	0·0125	82	1·7341	1·2737	0·7074	8·0
θ	0·0	0·3499	0·6560	0·9585	0·9987	1·0	—	—	0·0125	60	1·7341	1·2737	0·7074	4·0
θ	0·0	0·3497	0·6572	0·9607	0·9990	1·0	—	—	0·25	10	1·7146	1·2693	0·7108	4·0
θ	0·0	0·3499	0·6560	0·9585	0·9987	0·9999	1·0	1·0	0·0125	82	1·7341	1·2737	0·7074	8·0

† h_w = step size at the wall node $= \eta_{w+1} - \eta_w$

($f''_w = 1{\cdot}7338$; $g''_w = 1{\cdot}2786$; $\theta'_w = 0{\cdot}7076$; Reference 2)

Table II. Comparison of f''_w, g''_w and θ'_w with reference 1

K	0·5	0·0	1·0	−0·5	0·0	0·0	0·0	1·0	1·0
$\dfrac{H_w}{H_0}$	2·0	1·00	1·0	1·0	0·0	1·0	1·0	0·0	1·0
Pr	1·0	0·70	0·7	1·0	1·0	1·0	2·0	1·0	1·0
f''_w	1·7340	1·2327	1·3121	1·2295	0·6495	1·2327	1·7368	0·822	1·3121
g''_w	1·2736	0·5707	1·3121	−0·09177	0·5074	0·5707	0·6157	0·822	1·3121
θ'_w	0·7075	0·4975	0·6653	0·5464	0·5074	0·5707	0·6157	0·6989	0·7621
f''_w Reference 1	1·7338	1·1326	1·3119	1·2302	0·6489	1·2326	1·7362	0·8219	1·3119
g''_w Reference 1	1·2786	0·5705	1·3119	−0·115	0·5067	0·5705	0·6156	0·8219	1·3119
θ'_w Reference 1	0·7076	—	—	0·5484	0·5067	0·5705	0·6156	0·6989	0·7622

NOMENCLATURE

E_g = error of the quadrature formula in equation (12a)

E_f = error of the quadrature formula in equation (12b)

f, g = stream functions for similar solutions

H = enthalpy

K = ratio of principal velocity gradients $= \dfrac{\mathrm{d}W_e/\mathrm{d}z}{\mathrm{d}U_e/\mathrm{d}X_s}$

Pr = Prandtl number

U, W = transformed velocity in X and Z directions, respectively

X, Z = transformed co-ordinates along the body surfaces

Y = transformed normal co-ordinate

η = boundary layer similarity variable $= Y\sqrt{\left[\dfrac{(\mathrm{d}U_e/\mathrm{d}X)_s}{\nu_0}\right]}$

θ $= \dfrac{H - H_w}{H_0 - H_w}$

ν = kinematic viscosity

S_1 $= f(\eta)x\sqrt{\left[\nu_0\left\{\dfrac{\mathrm{d}U_e}{\mathrm{d}X}\right\}_s\right]}$

S_2 $= Kg(\eta)z\sqrt{\left[\nu_0\left\{\dfrac{\mathrm{d}U_e}{\mathrm{d}X}\right\}_s\right]}$

Ψ $= f'$

ϕ $= g'$

$(\ )_e$ = local condition outside the boundary layer

$(\ )_s$ = stagnation point value

$(\ )_w$ = wall value

$(\ )_0$ = free stream stagnation value

$(\ )_i$ = node index

$(\ )^k$ = iteration number

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